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Global Existence and Uniqueness Theorem for 3D – Navier-Stokes System on \mathbb{T}^3 for Small Initial Conditions in the Spaces $\Phi(\alpha)$.

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Dedicated to G.A. Margulis on the occasion of his sixtieth birthday.

Abstract: We consider Cauchy problem for three-dimensional Navier-Stokes system with periodic boundary conditions with initial data from the space of pseudo-measures $\Phi(\alpha)$. We provide global existence and uniqueness of the solution for sufficiently small initial data.

1. INTRODUCTION

Three-dimensional Navier-Stokes system with periodic boundary conditions after Fourier transform can be written in the form:

$$(1) \quad v(t, k) = \exp -t|k|^2 v_0(k) + 2\pi i \int_0^t \exp\{-(t-s)|k|^2\} \sum_{l \in \mathbb{Z}^3} \langle k, v(s, k-l) \rangle P_k v(s, l) ds$$

Here $k \in \mathbb{Z}^3$, $t \in \mathbb{R}_+$, $v(t, k) \in \mathbb{C}^3$, $v(t, k) \perp k$ for any $k \neq 0$ and $v(t, 0) = 0$ for all $t > 0$. $v_0(k)$ is the initial condition and P_k denotes Leray projector to the subspace orthogonal to k and has the form $P_k = \text{Id} - \frac{\langle k, \cdot \rangle}{|k|^2} k$. Also (??) assumes that the viscosity $\nu = 1$ and that the external forcing is absent.

T. Kato in [?] proved the local existence theorem for the 3D-Navier-Stokes system on \mathbb{R}^3 and global existence and uniqueness theorem in the space $L^{\frac{3}{2}}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ for small initial conditions.

In this paper we consider Cauchy problem for the system (??) with initial data from the space $\Phi(\alpha)$ which is analogous to the subspace $\Phi(\alpha, \alpha)$ introduced in [?], [?] and consists of functions of the form

$$\Phi(\alpha) = \left\{ f(k) = \frac{c(k)}{|k|^\alpha}, k \neq 0 \mid \sup_k |c(k)| < \infty \right\}, \quad \|f(k)\|_\alpha = \sup_{k \in \mathbb{Z}^3} |k|^\alpha |f(k)|$$

We assume $\alpha > 2$ and shall write $\alpha = 2 + \varepsilon$. V. Kaloshin and Yu. Sannikov announced the global existence theorem in the spaces $\Phi(\alpha)$, $\alpha \geq 2$ for small initial data (see [?]). In this paper we give a detailed proof of this result which shows also the character of decay of solutions in this case. It is worthwhile to mention that according to our point of view a similar result is not valid in the continuous case of $k \in \mathbb{R}^3$.

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2. MAIN RESULT

The purpose of this paper is to prove the following theorem.

THEOREM 1. *Let $0 < 3\varepsilon < 1$ and $\|v_0\|_\alpha \leq \delta$ where $v_0 = \frac{c_0(k)}{|k|^\alpha}$ is the initial condition and $\delta = \delta(\alpha)$ is sufficiently small. Then the equation (??) has a global solution $v(t, k) = \frac{c(t, k)}{|k|^\alpha}$ such that $c(t, k)$ is a continuous mapping of $[0, \infty)$ into $L^\infty(\mathbb{Z}^3 \setminus \{0\})$, $t > 0$.*

The proof of the Theorem ?? goes by induction. Put $H_0^{(0)}(k) = \frac{c_0(k)}{|k|^\alpha}$, $k \neq 0$, $H_0^{(1)}(k) = G_0(k) = 0$ and assume that for some integer m we constructed the solution $v(t, k)$, $0 \leq t \leq m$, such that

$$(2) \quad v(m, k) = H_m^{(0)}(k) + H_m^{(1)}(k) + G_m(k)$$

where

$$H_m^{(0)}(k) = \frac{\exp\{-m|k|^2\}c_0(k)}{|k|^\alpha},$$

$$H_m^{(1)}(k) = \sum_{j=1}^m \exp\{-(m-j)|k|^2\}h_j^{(1)}(k)$$

and

$$G_m(k) = \sum_{j=1}^m \exp\{-(m-j)|k|^2\}g_j(k)$$

Suppose that for all $j \leq m$ functions $h_j^{(1)}(k)$ satisfy the inequalities:

$$(3) \quad |h_j^{(1)}(k)| \leq \frac{D_1 \delta^2 \exp \left\{ -\frac{j}{2} |k|^2 \right\}}{|k|^{2\varepsilon}},$$

while the functions $g_j(k)$ satisfy the inequalities:

$$(4) \quad |g_j(k)| \leq \frac{D_2 \delta^2 \exp \{ -d_1 |k| \sqrt{m} \}}{|k|^\beta}$$

Here $\beta > 3$ is a constant and d, D with indices denote various absolute constants which appear during the proof, but their exact values play no role in the arguments.

Consider $0 \leq t \leq 1$ and write down the solution of (??) in the form:

$$(5) \quad \begin{aligned} v(t+m, k) = & \frac{\exp \{ -(m+t) |k|^2 \} c_0(k)}{|k|^\alpha} + \sum_{j=1}^m \frac{\exp \{ -(m-j+t) |k|^2 \} h_j^{(1)}(k)}{|k|^{2\varepsilon}} + \\ & + \frac{h_{m+1}^{(1)}(t, k)}{|k|^{2\varepsilon}} + \sum_{j=1}^m \exp \{ -(m-j+t) |k|^2 \} g_j(k) + g_{m+1}(t, k) \end{aligned}$$

We show that the inequalities (??), (??) holds for $h_{m+1}^{(1)}(1, k)$ and $g_{m+1}(1, k)$ respectively.

3. PROOF OF THE MAIN RESULT.

Denote

$$(6) \quad H_{m+1}^{(0)}(t, k) = \frac{\exp \{ -(m+t) |k|^2 \} c_0(k)}{|k|^\alpha},$$

$$(7) \quad H_{m+1}^{(1)}(t, k) = \sum_{j=1}^m \frac{\exp \{ -(m-j+t) |k|^2 \} h_j^{(1)}(t, k)}{|k|^{2\varepsilon}} + \frac{h_{m+1}^{(1)}(t, k)}{|k|^{2\varepsilon}},$$

$$(8) \quad G_{m+1}(t, k) = \sum_{j=1}^m \exp \{ -(m-j+t) |k|^2 \} g_j(t, k)$$

and

$$(H' \circledast H'')(t, k) = i \int_0^t \exp\{-(t-s)|k|^2\} \sum_{\substack{k \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\langle k, H'(s, k-l) \rangle P_k H''(s, l)}{|k-l|^\alpha |l|^\alpha}$$

If we substitute (??) into (??) we can write the expression for $h_{m+1}^{(1)}(t, k)$:

$$(9) \quad h_{m+1}^{(1)}(t, k) = |k|^{2\varepsilon} \left(H_{m+1}^{(0)} \circledast H_{m+1}^{(0)} \right)(t, k)$$

and the expression for $g_{m+1}(t, k)$:

$$(10) \quad g_{m+1}(t, k) = \sum_{j_1=1}^8 I_{m+1}^{(1, j_1)}(t, k) + \sum_{j_2=1}^3 I_{m+1}^{(2, j_2)}(t, k) + I_{m+1}^{(3)}(t, k)$$

where

$$\begin{aligned} I_{m+1}^{(1, j_1)}(t, k) &= (H' \circledast H'')(t, k), \\ I_{m+1}^{(2, j_2)}(t, k) &= (H' \circledast g_{m+1})(t, k) + (g_{m+1} \circledast H')(t, k) \end{aligned}$$

and H', H'' are either $H_{m+1}^{(0)}(t, k)$, or $H_{m+1}^{(1)}(t, k)$ or $G_{m+1}(t, k)$ except the case $H' = H'' = H_{m+1}^{(0)}$ which corresponds to the $h_{m+1}^{(1)}$ according to (??). Therefore j_1 changes from 1 to 8 and j_2 changes from 1 to 3. Also

$$I_{m+1}^{(3)}(t, k) = g_{m+1} \circledast g_{m+1}.$$

We see that $I_{m+1}^{(1)}$ does not depend on $g_{m+1}(t, k)$, $I_{m+1}^{(2)}$ is a linear function of $g_{m+1}(t, k)$ and $I_{m+1}^{(3)}$ is a quadratic function of $g_{m+1}(t, k)$. Therefore (??) is a typical equation which can be solved by iterations if the coefficients are small enough. Below we provide necessary estimates and later we come back to the analysis of (8).

3.1. First estimates. Here we show that all functions $h_{m+1}^{(1)}$ behaves like gaussian functions of $t|k|$ and then provide necessary estimates for coefficients in (??).

As in [?], [?], we use the identity:

$$(11) \quad a_1|k-l|^2 + a_2|l|^2 = \frac{a_1 a_2}{a_1 + a_2} |k|^2 + (a_1 + a_2) \left| l - \frac{a_1}{a_1 + a_2} k \right|^2$$

An estimate of $H_{m+1}^{(1)}$. At first we estimate $h_{m+1}^{(1)}(t, k)$. From (??) it follows that

$$h_{m+1}^{(1)}(t, k) = (H_{m+1}^{(0)} \circledast H_{m+1}^{(0)})(t, k) = 2\pi i \int_0^t \exp\{-(t-s)|k|^2\} \cdot \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\langle k, c_0(k-l) \rangle P_k c_0(l)}{|k-l|^\alpha |l|^\alpha} \exp\{-(m+s)|k-l|^2 - (m+s)|l|^2\} ds$$

Using (??) we can write

$$\begin{aligned} |h_{m+1}^{(1)}(t, k)| &\leq \exp\left\{-\frac{m|k|^2}{2}\right\} \int_0^t \exp\left\{-(t-\frac{s}{2})|k|^2\right\} \cdot \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\langle k, c_0(k-l) \rangle P_k c_0(l)}{|k-l|^\alpha |l|^\alpha} \exp\left\{-2m|l-\frac{1}{2}k|^2\right\} ds \leq \\ &\leq \delta^2 \exp\left\{-\frac{m|k|^2}{2}\right\} \frac{\exp\{-\frac{t}{2}|k|^2\} - \exp\{-t|k|^2\}}{|k|^2} \cdot \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\exp\{-2m|l-\frac{1}{2}k|^2\}}{|k-l|^\alpha |l|^\alpha} \leq \\ &\leq \frac{D_3 \delta^2}{|k|^{2\varepsilon}} \exp\left\{-\frac{(m+t)|k|^2}{2}\right\} \frac{1 - \exp\{-\frac{t}{2}|k|^2\}}{|k|^2} \end{aligned}$$

Substituting this inequality to (??) we conclude

$$\begin{aligned} (12) \quad \left| H_{m+1}^{(1)}(t, k) \right| &\leq \frac{D_3 \delta^2}{|k|^{2\varepsilon}} \frac{(1 - \exp\{-\frac{t}{2}|k|^2\})}{|k|^2} \sum_{j=1}^{m+1} \exp\left\{-(m+1-\frac{j}{2})|k|^2\right\} \leq \\ &\leq \frac{D_4 \delta^2}{|k|^{2\varepsilon}} \frac{(1 - \exp\{-\frac{t}{2}|k|^2\})}{|k|^2} \exp\left\{-\frac{(m+1)|k|^2}{2}\right\} \end{aligned}$$

Estimates for $H_{m+1}^{(j_1)} \circledast H_{m+1}^{(j_2)}$. We present detailed estimate only for $H_{m+1}^{(0)} \circledast H_{m+1}^{(1)}$ since all other terms can be estimated in the same manner. From (??) and

(??) we have

$$\begin{aligned}
& \left| (H_{m+1}^{(0)} \otimes H_{m+1}^{(1)})(t, k) \right| \leq |k| \delta^3 \int_0^t \exp\{-(t-s)|k|^2\} \cdot \\
& \quad \cdot \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\exp\{-(m+s)|l|^2 - \frac{m+s}{2}|k-l|^2\}}{|k-l|^{2\varepsilon}|l|^\alpha} ds \leq \\
& \leq |k| \delta^3 \exp\left\{-\frac{m+1}{3}|k|^2\right\} \int_0^t \exp\{-(t-s)|k|^2\} \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\exp\{-\frac{3}{2}(m+s)|l - \frac{m+s}{3}k|^2\}}{|k-l|^{2\varepsilon}|l|^\alpha} ds
\end{aligned}$$

Since $\alpha + 2\varepsilon > 2$ the last sum is not more than some constant D_5 . We get

$$(13) \quad \left| (H_{m+1}^{(0)} \otimes H_{m+1}^{(1)})(t, k) \right| \leq |k| D_5 \exp\left\{-\frac{m+1}{3}|k|^2\right\} \frac{1 - \exp\{-t|k|^2\}}{|k|^2}$$

Similarly, for $(H_{m+1}^{(1)} \otimes H_{m+1}^{(1)})(t, k)$ we can write

$$(14) \quad \left| (H_{m+1}^{(1)} \otimes H_{m+1}^{(1)})(t, k) \right| \leq D_6 \exp\left\{-\frac{m+1}{4}|k|^2\right\} \frac{1 - \exp\{-t|k|^2\}}{|k|}$$

3.2. Spaces $\mathcal{F}_m(c)$. Fix positive constant $\beta > 0$ and introduce functional space $\mathcal{F}_m(c)$

$$\mathcal{F}_m(c) = \left\{ f(k) \mid |f(k)| \leq \frac{D_7}{|k|^\beta} \exp\{-c\sqrt{m}|k|\}, k \neq 0 \right\}, \|f\|_{m,c} = \inf D_7$$

We show that functions $g_{m+1}(t, k)$ belong to the spaces \mathcal{F}_m with uniform constant if only all coefficients in (??) are sufficiently small. First of all we show that $H^{(j_1)} \otimes H^{(j_2)}$, $j_1 + j_2 > 0$ belongs to the space $\mathcal{F}_m(d_2)$ for some constant d_2 . It follows from previous estimates, that all of these functions decay as a Gaussian functions. For our purpose it is convenient to consider them as functions from the space $\mathcal{F}_m(d_2)$. Since $m|k| \geq 1$ we can write

$$\exp\left\{-\frac{m|k|^2}{3}\right\} \leq \frac{D_8}{|k|^\beta} \exp\left\{-\frac{\sqrt{m}|k|}{\sqrt{3}}\right\}$$

for some constant D_8 . We see that $(H_{m+1}^{(0)} \otimes H_{m+1}^{(1)})(t, k) \in \mathcal{F}_{m+1}(\frac{1}{\sqrt{3}})$ and

$$(15) \quad \|H_{m+1}^{(0)} \otimes H_{m+1}^{(1)}\|_{m+1, \frac{1}{\sqrt{3}}} \leq D_9$$

for some constant D_9 , which does not depend on t .

Assuming that $G_{m+1} \in \mathcal{F}_{m+1}(d_2)$ we can write for $G_{m+1} \circledast H_{m+1}^{(0)}$

$$|(G_{m+1} \circledast H_{m+1}^{(0)})(t, k)| \leq \|G_{m+1}\|_{m+1, d_2} \delta |k| \int_0^t \exp\{-(t-s)|k|^2\} \cdot \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\exp\{-d_2 \sqrt{m}|k-l| - m|l|^2\}}{|l|^\alpha |k-l|^{\beta_1}} ds$$

For the last expression we get

$$\begin{aligned} \exp\{-d_2 \sqrt{m}|k-l| - m|l|^2\} &\leq \exp\{-d_2 \sqrt{m}|k|\} \exp\{d_2 \sqrt{m}|l| - m|l|^2\} \leq \\ &\leq D_{10} \exp\{-d_2 \sqrt{m}|k|\} \exp\{|l - \frac{d_2}{2\sqrt{m}}|^2\} \end{aligned}$$

So for $(G_{m+1} \circledast H_{m+1}^{(0)})(t, k)$ we obtain

$$|(G_{m+1} \circledast H_{m+1}^{(0)})(t, k)| \leq D_{11} \|G_{m+1}\|_{m+1, d_2} \delta \frac{\exp\{-d_2 |k| \sqrt{m}\}}{|k|^{\beta_1}} \frac{1 - \exp\{-t|k|^2\}}{|k|^2}$$

All other terms in $I_{m+1}^{(1)}(t, k)$ can be similarly estimated.

Thus we embed the first term in the representation of $g_{m+1}(t, k)$ given by (??) into the space $\mathcal{F}_{m+1}(d_2)$. Now we provide the necessary estimates for the terms $I_{m+1}^{(3)}$ and $I_{m+1}^{(2)}$.

Estimate for $I_{m+1}^{(3)}$. We show, that for given functions $f_1, f_2 \in \mathcal{F}_{m+1}(d_2)$ $f_1 \circledast f_2$ also belongs to the space $\mathcal{F}_{m+1}(d_2)$ and $\|f_1, f_2\|_{m+1, d_2} \leq D_{12} \|f_1\|_{m+1, d_2} \|f_2\|_{m+1, d_2} m +$ for some constant D_{12} .

Write down the estimate

$$\begin{aligned} |f_1 \circledast f_2| &\leq \|f_1\|_{m+1, d_2} \|f_2\|_{m+1, d_2} |k| \int_0^t \exp\{-(t-s)|k|^2\} \cdot \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\exp\{-d_2 \sqrt{m+1}(|l| - |k-l|)\}}{|l|^\beta |k-l|^\beta} ds \leq \\ &\leq \frac{D_{13} \|f_1\|_{m+1, d_2} \|f_2\|_{m+1, d_2}}{|k|^{2\beta-3}} \exp\{-d_2 |k| \sqrt{m+1}\} \frac{1 - \exp\{-t|k|^2\}}{|k|} \end{aligned}$$

Since $\beta > 3$ and the last expression is not more than 1, we get

$$(16) \quad |f_1 \circledast f_2| \leq \frac{D_{14} \|f_1\|_{m+1, d_2} \|f_2\|_{m+1, d_2}}{|k|^\beta} \exp\{-d_2 \sqrt{m+1}|k|\}$$

In particular, for $g_{m+1}(t, k) \in \mathcal{F}_{m+1}(d_2)$ it follows that

$$(17) \quad \|I_{m+1}^{(3)}(t, k)\|_{m+1, d_2} \leq D_{14} \|g_{m+1}(t, k)\|_{m+1, d_2}^2$$

Estimates for $I_{m+1}^{(2)}$. Here we produce the upper bound for $\|I^{(2)}(t, k)\|_{m+1, D_{14}} = \sum_{j_2=1}^3 I_{m+1}^{(2, j_2)}(t, k)$ assuming, that $g_{m+1}(t, k) \in \mathcal{F}_{m+1}(d_2)$.

$$\begin{aligned} |(g_{m+1}(t, k) \otimes H_{m+1}^{(0)}(t, k))| &\leq \|g_{m+1}\|_{m+1, d_2} \delta |k| \int_0^t \exp\{-(t-s)|k|^2\} \cdot \\ &\cdot \sum_{\substack{l \in \mathbb{Z}^3 \setminus \{0\} \\ k-l \neq 0}} \frac{\exp\{-d_2 \sqrt{m}|k-l| - m|l|^2\}}{|k-l|^\beta |l|^\alpha} ds \end{aligned}$$

Again for the last expression holds

$$\exp\{-d_2 \sqrt{m}|k-l| - m|l|^2\} \leq D_{15} \exp\{-d_2 \sqrt{m}|k|\} \exp\{|l - \frac{d_2}{2\sqrt{m}}|^2\}$$

So for $(g_{m+1}(t, k) \otimes H_{m+1}^{(0)}(t, k))$ we can write

$$|(g_{m+1}(t, k) \otimes H_{m+1}^{(0)}(t, k))| \leq D_{16} \|g_{m+1}\|_{m+1, d_2} \delta \frac{1 - \exp\{-t|k|^2\}}{|k|} \frac{\exp\{-d_2 \sqrt{m}|k|\}}{|k|^\beta}$$

For the terms $g_{m+1} \otimes G_{m+1}$ we can produce an appropriate estimate using (??). All other terms in $I_{m+1}^{(2)}(t, k)$ can be estimated in a similar way.

Collecting all present estimates we see that for some constant *newcon*

$$(18) \quad \|g_{m+1}(t, k)\|_{m+1, d_2} \leq D_{17} \delta^2 + D_{18} \delta \|g_{m+1}\|_{m+1, d_2} + D_{19} \|g_{m+1}\|_{m+1, d_2}^2$$

So for sufficiently small δ all coefficients in (??) are small and the equation (??) can be solved by iterations. The solution $g_{m+1}(t, k)$ belongs to $\mathcal{F}_{m+1}(d_2)$ and unique in this class of functions. Each function $g_{m+1}(t, k)$ provides the unique solution $v(m+t, k)$ of (??). The Theorem ?? is proven.

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